# LINEAR ALGEBRA

## Week 4

**Proposition.** Two $m\times n$ matrices $A$ and $B$ are row-equivalent iff $B=PA$ for some invertible $m\times n$ matrix $P$.

**Proof.** "$\Rightarrow$": $A\sim B$ implies $B=\varepsilon(A)$ and hence $\left[A|I\right]\longrightarrow\left[B|P\right]$ with the invertible $P=\varepsilon(I)$, and $B=\varepsilon(A)=\varepsilon(I).A=PA$.

"$\Leftarrow$": Let $B=PA$ and $P$ is invertible. By the previous theorem, $P=\varepsilon(I)$. Then $B=\varepsilon(I).A=\varepsilon(A)$ and hence $B\sim A$.

**Exercise 1.** Let $A, B, C$ and $D$ be $3\times 4-$matrices such that

$$A\xrightarrow{-R_1+R_3}B\xrightarrow{R_1\leftrightarrow R_2}D$$

and

$$C\xrightarrow{3R_2+R_3}D.$$

Find an invertible matrix $P$ such that $PA=C$ and write $P$ as a product of 3 elementary matrices accordingly to the diagrams above.

**Solution** $A\xrightarrow[\varepsilon_1]{-R_1+R_3}B\xrightarrow[\varepsilon_2]{R_1\leftrightarrow R_2}D\xrightarrow[\varepsilon_3]{-3R_2+R_3}C$

$$P=\varepsilon_3\varepsilon_2\varepsilon_1(I)=\varepsilon_3(I)\varepsilon_2(I)\varepsilon_1(I)=\begin{bmatrix}1&0&0\\0&1&0\\0&-3&1\end{bmatrix}.\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}.\begin{bmatrix}1&0&0\\0&1&0\\-1&0&1\end{bmatrix}=\varepsilon_3.\begin{bmatrix}0&1&0\\1&0&0\\-1&0&1\end{bmatrix}$$

$$=\begin{bmatrix}1&0&0\\0&1&0\\0&-3&1\end{bmatrix}.\begin{bmatrix}0&1&0\\1&0&0\\-1&0&1\end{bmatrix}=\begin{bmatrix}0&1&0\\1&0&0\\-4&0&1\end{bmatrix}=P$$

$$\left(\begin{array}{l} I=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\Rightarrow \varepsilon_3=-3R_2+R_3\Rightarrow \varepsilon_3(I)=\begin{bmatrix}1&0&0\\0&1&0\\0&-3&1\end{bmatrix}\\ \varepsilon_2=R_1\leftrightarrow R_2\Rightarrow \varepsilon_2(I)=\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}\\ \varepsilon_1=-R_1+R_3\Rightarrow \varepsilon_1(I)=\begin{bmatrix}1&0&0\\0&1&0\\-1&0&1\end{bmatrix}\end{array}\right)$$

**Exercise 2.** Let $C,$ D, L, M and $K$ be $2\times 4-$matrices such that

$C\xrightarrow{R_1\leftrightarrow R_2}L\xrightarrow{2R_2}K$ and $D\xrightarrow{2R_2+R_1}M\xrightarrow{3R_1}K$.

Find an invertible matrix $P$ such that $PC=D$ and write $P$ as a product of 4 elementary matrices accordingly to the diagrams above.

**Solution.** $C\xrightarrow[\varepsilon_1]{R_1\leftrightarrow R_2}L\xrightarrow[\varepsilon_2]{2R_2}K\xrightarrow[\varepsilon_3]{\frac{1}{3}R_1}M\xrightarrow[\varepsilon_4]{-2R_2+R_1}D$

$$P=\varepsilon_4\varepsilon_3\varepsilon_2\varepsilon_1(I)=\begin{bmatrix}1&-2\\0&1\end{bmatrix}\cdot\begin{bmatrix}1/3&0\\0&1\end{bmatrix}\cdot\begin{bmatrix}1&0\\0&2\end{bmatrix}\cdot\begin{bmatrix}0&1\\1&0\end{bmatrix}=\begin{bmatrix}1&-2\\0&1\end{bmatrix}\cdot\begin{bmatrix}1/3&0\\0&1\end{bmatrix}\cdot\begin{bmatrix}0&1\\2&0\end{bmatrix}$$

$$=\begin{bmatrix}1&-2\\0&1\end{bmatrix}\cdot\begin{bmatrix}0&1/3\\2&0\end{bmatrix}=\begin{bmatrix}-4&1/3\\2&0\end{bmatrix}$$

**Systems of Linear Equations**

A system

$$\begin{cases}a_{11}x_1+a_{12}x_2+\cdots+a_{1n}x_n=b_1\\a_{21}x_1+a_{22}x_2+\cdots+a_{2n}x_n=b_2\\\cdots\cdots\cdots\cdots\cdots\cdots\\a_{m1}x_1+a_{m2}x_2+\cdots+a_{mn}x_n=b_m\end{cases}$$

is called a system of $m$ linear equations in $n$ unknowns (variables) $x_1,\ x_2,\dots,x_n$ or a linear system.

In the matrix form: $A_{m\times n}.X_{n\times 1}=B_{m\times 1}$, where

$$A=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$ is the **coefficient matrix**,

$$X=\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$ is the **solution matrix**, and

$$B=\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$ is the **right part** of the system.

The collection of all solutions of a linear system is called the **solution set** of the system.

$$AX=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}=\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}.$$

**Example 1.** $\begin{cases} 2x_1+x_2-x_3=1 \\ 3x_1-2x_2+4x_3=2 \\ x_1+x_2+5x_3=3 \end{cases}$

$$A=\begin{bmatrix} 2 & 1 & -1 \\ 3 & -2 & 4 \\ 1 & 1 & 5 \end{bmatrix}$$ is coeffient matrix of the system.

$$X=\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$ is the solution matrix.

$$B=\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$ is the right part of the system.

**Definition 1.** A system $AX=B$ which has at least one solution is called **consistent**. If it has no solutions, then $AX=B$ is called **inconsistent**.

It will be shown below that every consistent linear system has either a unique solution or infinitely many solutions.

**Example 2.** Clearly, $\begin{cases} x+y=1 \\ 2x+2y=6 \end{cases}$ is inconsistent.

**Definition 2.** Consider a linear system $A_{m\times n}.X_{n\times 1}=B_{m\times 1}$. The $m\times(n+1)$ matrix $\left[A\middle|B\right]$ is called the **augmented matrix** of the system.

**Example 3**. Find the augmented matrix of the system $\begin{cases} 2x_1+x_2-x_3=1 \\ 3x_1-2x_2+4x_3=2 \\ x_1+x_2+5x_3=3 \end{cases}$.

$$\left[A\middle|B\right]=\left[\begin{array}{ccc|c} 2 & 1 & -1 & 1 \\ 3 & -2 & 4 & 2 \\ 1 & 1 & 5 & 3 \end{array}\right]_{3\times 4}.$$

**Definition 3.** Two linear systems are said to be **equivalent** if they have row-equivalent augmented matrices.

**Theorem 1.** Let $A_{m\times n}.X_{n\times 1}=B_{m\times 1}$ and $A'_{m\times n}.X_{n\times 1}=B'_{m\times 1}$ be equivalent systems. Then both systems have the same solution sets.

**Proof.** Since $\left[A\middle|B\right]\sim\left[A'\middle|B'\right]$, there exists an invertible $P_{m\times m}$ with $\left[A'\middle|B'\right]=P.\left[A\middle|B\right]$. Then

$$AX=B\Leftrightarrow \underbrace{P.A}_{A'}X=\underbrace{P.B}_{B'}\Leftrightarrow A'X=B'.$$

**Example 4.** Linear systems possessing the same solution sets need not to be equivalent. For example, two systems: $\begin{cases} x+y=1 \\ x+y=2 \end{cases}$ and $\begin{cases} x-y=1 \\ x-y=2 \end{cases}$ are both inconsistent, so they have the same empty set of solutions, yet

$$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 1 & 1 & 2 \end{array}\right]\sim\underbrace{\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 0 & 1 \end{array}\right]}_{RREM}\nsim\underbrace{\left[\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & -2 \end{array}\right]}_{RREM}\sim\left[\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & -1 & 2 \end{array}\right]\sim\left[\begin{array}{cc|c} 1 & -1 & 1 \\ 0 & -1 & 2 \end{array}\right]$$

**Solving Methods for Linear Equation Systems**

There are two main methods for solving systems of linear equations:

**1)** Gaussian elimination is transforming $\left[A|B\right]$ to an echelon matrix $\left[A'|B'\right]$.

**2)** Gauss – Jordan reduction is transforming $\left[A|B\right]$ to RREM $\left[A'|B'\right]$.

It should be clear that: if $\left[A|B\right]\to\left[A'|B'\right]$, and the echelon matrix $\left[A'|B'\right]$ has a leading entry in the last column, then the system $A'X = B'$, and hence $AX = B$, is inconsistent. Otherwise, $A'X = B'$, and hence $AX = B$, possesses solution(s) which can be obtained by back substitution. A **pivot** of a matrix is the coordinates (indexes) of an entry of the echelon form of the matrix. A **pivot column (pivot row)** of a matrix is a column (row) of the matrix that contains an entry indexed by a pivot.

**Example 5.** Solve the system $\begin{cases} x+y-z+t-u=1 \\ y-t+u=2 \\ z-u=1 \end{cases}$.

**Solution** $\left[A|B\right]=\left[\begin{array}{ccccc|c} \boxed{1} & 1 & -1 & 1 & -1 & 1 \\ 0 & \boxed{1} & 0 & -1 & 1 & 2 \\ 0 & 0 & \boxed{1} & 0 & -1 & 1 \end{array}\right]$ is already an echolon matrix. Since there is no leading entry in the last column, $AX = B$ is consistent.

By the back substitution:

$$\begin{aligned} &z=1+u \\ &y=2+t-u \\ &x=1-y+z-t+u=1-(2+t-u)+1+u-t+u=-2t+3u \end{aligned}$$

Or, shortly, in the matrix form:

$$X=\begin{bmatrix} x \\ y \\ z \\ t \\ u \end{bmatrix}=\begin{bmatrix} -2t+3u \\ 2+t-u \\ 1+u \\ t \\ u \end{bmatrix}=\begin{bmatrix} 0 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}+t\begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}+u\begin{bmatrix} 3 \\ -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}.$$

**Definition 4.** We call a variable of a system $AX = B$:

**1) basic** if it corresponds to a pivot column of $\left[A|B\right]$;

**2) free** if it does do not correspond to any pivot column of $\left[A|B\right]$.

**Example 6.**

$$\begin{array}{cc} & \begin{array}{ccccc} x & y & z & \boxed{t} & \boxed{u} \end{array} \\ \left[A|B\right]= & \left[\begin{array}{ccccc|c} \boxed{1} & 1 & -1 & 1 & -1 & 1 \\ 0 & \boxed{1} & 0 & -1 & 1 & 2 \\ 0 & 0 & \boxed{1} & 0 & -1 & 1 \end{array}\right] \end{array}$$

Hence $x$, $y$ and $z$ are basic, $t$ and $u$ are free variables.

**Definition 5.** Solution of a system $AX=B$ expressed in terms of free variables is called the **general solution**.

**Example 7.** In previous two examles, $X=\begin{bmatrix}0\\2\\1\\0\\0\end{bmatrix}+t\begin{bmatrix}-2\\1\\0\\1\\0\end{bmatrix}+u\begin{bmatrix}3\\-1\\1\\0\\1\end{bmatrix}$ is the general solution of $AX=B$. Any partial solution of the $AX=B$ can be obtained by appropriate choice of values of free variables t and u. For example, $X_b=\begin{bmatrix}0\\2\\1\\0\\0\end{bmatrix}$ corresponds to the choice: $t=0$ and $u=0$.

$$X=\begin{bmatrix}0\\2\\1\\0\\0\end{bmatrix}+\begin{bmatrix}-2\\1\\0\\1\\0\end{bmatrix}+\begin{bmatrix}3\\-1\\1\\0\\1\end{bmatrix}=\begin{bmatrix}1\\2\\2\\1\\1\end{bmatrix}$$ corresponds to the choise: $t=1$ and $u=1$.

**Remarks 1.** For a consistent system $AX=B$, the number of basic variables is equal to the number of pivot columns of $A$. A consistent system $AX=B$ has a unique solution iff $AX=B$ has no free variables. It can be shown that if two consistent systems $A'_{m\times n}.X=B'_{m\times 1}$

and $A''_{m\times n}.X_{n\times 1} = B''_{m\times 1}$ have the same solution set, then $\left[A'\middle|B'\right] \sim \left[A''\middle|B''\right]$. It follows from the observation that the solution set of a system $AX = B$ such that $\left[A\middle|B\right]$ is RREM, in which the last column is not a pivot column, determines $\left[A\middle|B\right]$ uniquely.

**Exercises**

**1)** Find the general solution of the following system

$$\begin{aligned} x_1 + 2x_2 - 3x_4 + x_5 &= 2 \\ x_3 + 4x_4 - 2x_5 &= -1. \\ x_1 + 2x_2 + x_3 + x_4 - x_5 &= 1 \end{aligned}$$

**Solution.**

$$\left[\begin{array}{ccccc|c} 1 & 2 & 0 & -3 & 1 & 2 \\ 0 & 0 & 1 & 4 & -2 & -1 \\ 1 & 2 & 1 & 1 & -1 & 1 \end{array}\right] \xrightarrow{-R_1+R_3} \left[\begin{array}{ccccc|c} 1 & 2 & 0 & -3 & 1 & 2 \\ 0 & 0 & 1 & 4 & -2 & -1 \\ 0 & 0 & 1 & 4 & -2 & -1 \end{array}\right] \xrightarrow{-R_2+R_3} \left[\begin{array}{ccccc|c} \boxed{1} & 2 & 0 & -3 & 1 & 2 \\ 0 & 0 & \boxed{1} & 4 & -2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$x_1$ and $x_3$ are basics, $x_2, x_4$ and $x_5$ are free variables. And $\begin{cases} x_1 = 2 - 2x_2 + 3x_4 - x_5 \\ x_3 = -1 - 4x_4 + 2x_5 \end{cases}$. So the general solution is

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2 - 2x_2 + 3x_4 - x_5 \\ x_2 \\ -1 - 4x_4 + 2x_5 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ -1 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 3 \\ 0 \\ -4 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -1 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

**2)** Find the general solution of the following system

$$\begin{aligned} x + 2y + 3z &= 9 \\ 2x - y + z &= 8. \\ 3x - z &= 3 \end{aligned}$$

**Solution.**

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 2 & -1 & 1 & 8 \\ 3 & 0 & -1 & 3 \end{array}\right] \xrightarrow[-3R_1+R_3]{-2R_1+R_2} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 0 & -5 & -5 & -10 \\ 0 & -6 & -10 & -24 \end{array}\right] \xrightarrow[-\frac{1}{2}R_3]{-\frac{1}{5}R_2} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 0 & 1 & 1 & 2 \\ 0 & 3 & 5 & 12 \end{array}\right]$$

$$\xrightarrow{-3R_2+R_3} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 2 & 6 \end{array}\right] \xrightarrow{\frac{1}{2}R_3} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$$\left.\begin{array}{l} x+2y+3z=9 \\ y+z=2 \\ z=3 \end{array}\right\} \Rightarrow y=-3+2=-1, x=-2y-3z+9=2-9+9=2 \Leftarrow X=\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$$

**3)** Find the general solution of the following system

$$\begin{aligned} x_1+2x_2+3x_3+4x_4&=5 \\ x_2+2x_3+3x_4&=6. \\ x_2+2x_3+3x_4&=7 \end{aligned}$$

**Solution.**

$$\left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 2 & 3 & 6 \\ 0 & 1 & 2 & 3 & 7 \end{array}\right] \xrightarrow{-R_2+R_3} \left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 2 & 3 & 6 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

There is no solution because of the last row.

**4)** Find value(s) of $a \in \mathbb{R}$ for which the system

$$\begin{cases} x+y=3 \\ x+\left(a^2-8\right)y=a \end{cases}$$

**a)** has no solution;

**b)** has a unique solution;

**c)** has infinitely many solutions.

**Solution.**

$$\left[\begin{array}{cc|c} 1 & 1 & 3 \\ 1 & a^2-8 & a \end{array}\right] \xrightarrow{-R_1+R_2} \left[\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & a^2-9 & a-3 \end{array}\right]$$

a) $a^2-9=(a-3)(a+3)=0 \Rightarrow$ if $a=-3$ there is no solution since $0 \neq -6$;

b) if $a \neq \pm 3$ there is a unique solution;

c) if $a=3$ there are infinitely many solutions.

**5)**

$$\begin{cases} x+2y-3z=4 \\ 3x-y+5z=2 \\ 4x+y+\left(k^2-14\right)z=k+2 \end{cases}$$

For the linear system, which is given above, determine all values of $k$ for which the resulting linear system has

a) no solution;

b) a unique solution;

c) infinitely many solutions.

**6)** Find value(s) of $a, b \in \mathbb{R}$ for which the system

$$\begin{cases} 2x - y + 2az + t = b \\ -2x + ay - 3z = 4 \\ 2x - y + (2a+1)z + (a+1)t = 0 \\ -2x + y + (1-a)z - 2t = -2b - 2 \end{cases}$$

**a)** has no solution;

**b)** has a unique solution;

**c)** has infinitely many solutions.